SOME PROPERTIES OF LATTICE SUBSTITUTION SYSTEMS

DIRK FRETTLÖH

Abstract. If a partition of a lattice in $\mathbb{R}^d$ is selfsimilar, it is called lattice substitution system. Such sets represent nonperiodic, but highly ordered structures. An important property of such structures is, whether they are model sets or not. We give two sufficient conditions to decide, whether such a structure does not consist of regular model sets.

1. Introduction

A central property of strongly ordered, but aperiodic structures is the ability to show perfect pure point diffraction. Besides the well–known 'classical' examples which serve as models for physical quasicrystals, like the pentagonal Penrose tilings [Pen], the octagonal Ammann–Beenker tilings [AGS] and the icosahedral tilings of Kramer [Kra, KN] or Danzer [Dan1], there are a lot of other kinds of such structures, which may also show pure point diffraction. Examples are other tilings or point sets generated by a substitution or inflation, but also e.g. the visible points of a lattice [BMP], or so called 'deformed model sets' [BD], or generalized Kolakoski sequences Kol(p,q) [BS, Sing]. One class which is understood quite well consists of structures which arise from substitutions and which 'live on a lattice', i.e., the translation module of such a structure is a lattice. Examples are the chair tilings, the sphinx tilings or the Kolakoski sequence Kol(4,2).

Usually these structures are described either in terms of tilings or in terms of discrete (maybe coloured) point sets. The known conditions to prove pure point diffractivity usually are working with discrete point sets. To prove that, e.g., the chair tiling is pure point diffractive, one will construct a discrete point set D which is mutual locally derivable from the tiling, and apply the conditions to D. A more general way to describe D is as a Delone multiset [LMS1]. A Delone multiset $\boldsymbol{V}$ is a tupel $(V_1, \ldots, V_m)$, such that every V_i is a Delone set and $\bigcup_{i=1}^m V_i$ is a Delone set. A Delone multiset can be regarded as a coloured Delone set (cf. chapter 3). If $\bigcup_{i=1}^m V_i$ is a lattice, and $\boldsymbol{V}$ can be described by a substitution, $\boldsymbol{V}$ is called a lattice substitution system.

A model set is a discrete point set which can be obtained by the cut–and–project–method [M, M2]. By a theorem of Schlottmann [Sch], every regular model set is pure point diffractive.

In [LM] and [LMS2], Lee, Moody and Solomyak give several necessary and sufficient conditions on a lattice substitution system $\boldsymbol{V}$ to decide whether $\boldsymbol{V}$ consists of regular model sets (i.e., the V_i are model sets), and showed — in this case — that $\mathbf{V}$ is a model set if and only if $\mathbf{V}$ is pure point diffractive. Their conditions give an algorithm which can answer this questions in principle. If $\boldsymbol{V}$ consists of model sets, the algorithm terminates after finitely many steps and gives an affirmative answer. The number of steps may be large. If $\boldsymbol{V}$ does not consist of model sets, the algorithm does not terminate.

The aim of this paper is to give two sufficient conditions on a lattice substitution system $\boldsymbol{V}$ for not consisting of model sets. Both conditions can decide that a given $\boldsymbol{V}$ is not a model set in finite time, the time depending only on the inflation factor and the number of colours.

This article is organized as follows: In chapter 2 we give some basic definitions about tilings and mention some results we need later. Many of them are well known, but scattered over the literature (see [BG] for a list of references). For the convenience of the reader we therefore recall the results we need here.

Chapter 3 describes the construction of Delone multisets out of tilings and which properties transfer from tilings to Delone multisets.

In chapter 4 we recall the theorem of Lee, Moody and Solomyak, fitted to our setting, and prove two sufficient conditions to rule out the possibility that a given lattice substitution system can consist of model sets.

Throughout the text, $\mathbb{N}$ denotes the set of positive integers, $\mathrm{int}(A)$ the interior of A and $\mathrm{cl}(A)$ the closure of A. $\mathbb{B}^d$ denotes the d–dimensional unit ball $\{x \in \mathbb{R}^d \,|\, \|x\| \le 1\}$. $\#M$ denotes the cardinality of the set M.

2. Tilings and Inflation

A compact set $T \subseteq \mathbb{R}^d$ is called a *tile*, if $\emptyset \neq \mathrm{cl}(\mathrm{int}(T)) = T$. A *tiling* $\mathcal{T} = \{T_1, T_2, \ldots\}$ in $\mathbb{R}^d$ is a countable set of tiles, which fulfils $\mathrm{int}(T_i) \cap \mathrm{int}(T_j) = \emptyset$ for every $i \neq j$ (i.e., the tiles do not overlap) and $\mathbb{R}^d = \bigcup_{i \in \mathbb{N}} T_i$ (i.e., the tiles cover $\mathbb{R}^d$). If $\mathcal{F}$ is a set of tiles, such that every tile in $\mathcal{T}$ is a translate of some $T \in \mathcal{F}$, $\mathcal{F}$ is called *protoset* of $\mathcal{T}$, and the elements of $\mathcal{F}$ are called *prototiles*.

A *cluster* in $\mathcal{T}$ is a finite subset of $\mathcal{T}$. The set

$$\mathcal{C}_r(x, \mathcal{T}) := \{T \in \mathcal{T} \,|\, T \cap (x + r\mathbb{B}^d) \neq \emptyset\},$$

where $r > 0$, is called *r–cluster (in x)*. We also need the case $r = 0$, therefore we set

$$\mathcal{C}_0(x, \mathcal{T}) := \{T \in \mathcal{T} \,|\, x \in T\},$$

In the rest of the paper we will identify two clusters $\mathcal{C}, \mathcal{D}$ in two different ways. $\mathcal{C}$ and $\mathcal{D}$ are equal, if $\mathcal{C} = \mathcal{D}$, i.e., every tile $S \in \mathcal{C}$ has a corresponding tile $T \in \mathcal{D}$ such that $S = T$ and vice versa. $\mathcal{C}$ and $\mathcal{D}$ are *of the same type* if they are translation equivalent, i.e., there exists $t \in \mathbb{R}^d$ such that $\mathcal{D} = \mathcal{C} + t := \{T + t \,|\, T \in \mathcal{C}\}$. The analogous distinction is made for tiles.

A tiling $\mathcal{T}$ is called *nonperiodic*, if the only solution of $\mathcal{T} + x = \mathcal{T}$ is $x = 0$. A simple way to construct nonperiodic tilings is the *inflation* method. Given is a protoset $\mathcal{F} = \{T_1, \ldots, T_m\}$, a factor $\eta > 1$ (the *inflation factor*) and a rule, how to dissect $\eta T_1, \ldots, \eta T_m$ into tiles congruent to $T_1, \ldots, T_m$. This rule is given by a set $\mathcal{Q} = \{\mathcal{C}_1, \ldots, \mathcal{C}_m\}$ of clusters $\mathcal{C}_i = \{T_{i_j} + t_{i_j} \,|\, T_{i_j} \in \mathcal{F}, t_{i_j} \in \mathbb{R}^d\}$ where $\eta T_i = \bigcup_{T \in \mathcal{C}_i} T$. Notice that by the protoset $\mathcal{F} = \{T_1, \ldots, T_m\}$ we can describe tilings as well as clusters in the form $\{T_{i_1} + t_1, T_{i_2} + t_2, \ldots\} = \{T_{i_j} + t_j\}_{j \in J}$.

Figure 2 shows an example of such an inflation with factor 2. There are only two prototiles up to isometry, but since we identify tiles up to translation, the shown protoset consists of 16 tiles, 8 large ones and 8 small ones.

Every such inflation gives rise to an *inflation operator* infl. Let M be the set of all (finite or infinite) sets $\{T_{i_j} + t_j\}_{j \in J}$ of translates of the prototiles.

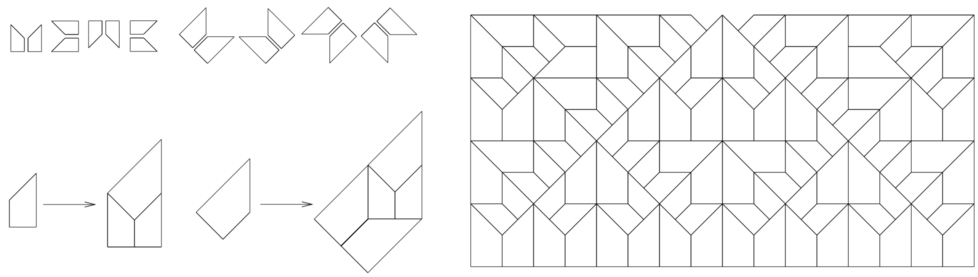

FIGURE 1. Example of an inflation: The protoset (top left), the inflation rule (bottom left) and a cluster of a tiling generated by this inflation (right). The inflation rule is shown for just one small and one large prototile, the other prototiles are inflated in the analogous way.

$$\mathrm{infl} : \mathsf{M} \to \mathsf{M}, \ \mathrm{infl}(\{T_{i_1} + t_1, T_{i_2} + t_2, \dots\}) = \{\mathcal{C}_{i_1} + \eta t_1, \mathcal{C}_{i_2} + \eta t_2, \dots\}$$

Obviously, infl can be iterated. Let $\mathrm{infl}^k := \mathrm{infl} \circ \mathrm{infl}^{k-1}$ and $\mathrm{infl}^1 := \mathrm{infl}$. Starting with a single tile $T_i + t_i$, we get by applying infl the *supertiles* $\mathrm{infl}(T_i + t_i), \mathrm{infl}^2(T_i + t_i), \dots, \mathrm{infl}^k(T_i + t_i), \dots$ of *first, second, ..., k-th ,... order.* These iterations fill larger and larger regions of space. Intuitively, one may say that this process leads to a tiling in the limit. Let us introduce a proper way to describe this:

Definition 2.1. *Let $(\mathcal{F}, \eta, \mathcal{Q})$ be an inflation, with* infl *the corresponding inflation operator. A tiling $\mathcal{T}$ is called an* inflation tiling (with operator infl) *if every cluster of $\mathcal{T}$ is a translate of a cluster in some supertile* $\mathrm{infl}^k(T_i)$, $T_i \in \mathcal{F}$.
The set of all these inflation tilings is called the inflation species*, written* $\mathbf{S}(\mathcal{F}, \eta, \mathcal{Q})$*,* $\mathbf{S}(\mathcal{F}, \mathrm{infl})$ *or simply* $\mathbf{S}$*.*
The $m \times m$ matrix (a_{ij}), $a_{ij} := \#\{T \in \mathcal{C}_j \mid T = T_i + t, t \in \mathbb{R}^d\}$ is called inflation matrix *of* $\mathbf{S}$.

In the following, species (resp. inflation species) are considered instead of single tilings. Therefore, 'a cluster of $\mathbf{S}$' is to be read as 'a cluster of some $\mathcal{T} \in \mathbf{S}$'.

A species is called *primitive* if the inflation matrix M is primitive, i.e., if there is a $k \geq 0$ such that all entries of M^k (and therefore all entries of M^ℓ, where $\ell > k$) are positive. Equivalently, a species is primitive, if for every $T \in \mathcal{F}$ the k–th order supertile contains all types of prototiles.

Theorem 2.1. *For every $k \in \mathbb{N}$ and every inflation species $\mathbf{S}(\mathcal{F}, \mathrm{infl})$ holds: $\mathbf{S}(\mathcal{F}, \mathrm{infl}) = \mathbf{S}(\mathcal{F}, \mathrm{infl}^k)$.*

Proof. We will show this only for primitive inflation species $\mathbf{S}(\mathcal{F}, \mathrm{infl})$ because that is all we need for this article. For the — much longer — complete proof see [FRE].

Every cluster of $\mathcal{T} \in \mathbf{S}(\mathcal{F}, \mathrm{infl}^k)$ occurs in some supertile $(\mathrm{infl}^k)^\ell(T_i)$ of ℓ–th order in $\mathbf{S}(\mathcal{F}, \mathrm{infl}^k)$, hence in a supertile of $k\ell$–th order in $\mathbf{S}(\mathcal{F}, \mathrm{infl})$. Therefore $\mathbf{S}(\mathcal{F}, \mathrm{infl}) \supseteq \mathbf{S}(\mathcal{F}, \mathrm{infl}^k)$.

Because $\mathbf{S}(\mathcal{F}, \mathrm{infl})$ is primitive, there is an $n \in \mathbb{N}$, such that every tile type $T \in \mathcal{F}$ occurs in every supertile $\mathrm{infl}^\ell(T_j), \ell \geq n$. Hence every supertile $\mathrm{infl}^s(T_i)$ occurs in every supertile $\mathrm{infl}^\ell(T_j), \ell \geq n + s, 1 \leq j \leq m$. In particular, there is a multiple of k with $r_s k \geq n + s$, therefore every supertile $\mathrm{infl}^s(T_i)$ occurs in every supertile $\mathrm{infl}^{r_s k}(T_j), 1 \leq j \leq m$.

Every cluster of $\mathbf{S}(\mathcal{F}, \text{infl})$ occurs in some supertile $\text{infl}^s(T_i)$, therefore in a supertile $\text{infl}^{r_s k}(T_j)$. So every cluster of $\mathbf{S}(\mathcal{F}, \text{infl})$ is a cluster of $\mathbf{S}(\mathcal{F}, \text{infl}^k)$. $\square$

Definition 2.2. *A species* $\mathbf{S}$ *is of* finite local complexity*, if for every* $r > 0$ *the number of types of* r*–clusters in* $\mathbf{S}$ *is finite.*
A species $\mathbf{S}$ *is* weakly repetitive*, if for every cluster* $\mathcal{C}$ *in* $\mathbf{S}$ *exists* $R_{\mathcal{C}} > 0$ *such that for all* $x \in \mathbb{R}^d$ *a translate of* $\mathcal{C}$ *occurs in every* $R_{\mathcal{C}}$*–cluster* $\mathcal{C}_{R_{\mathcal{C}}}(x, \mathcal{T}')$ *for all* $\mathcal{T}' \in \mathbf{S}$*.*
A species $\mathbf{S}$ *is* repetitive*, if for every* $r > 0$ *exists* $R_r > 0$ *such that for all* $x, y \in \mathbb{R}^d$ *a translate of* $\mathcal{C}_r(y, \mathcal{T})$ *occurs in every cluster* $\mathcal{C}_{R_r}(x, \mathcal{T}')$ *for all* $\mathcal{T}' \in \mathbf{S}$*.*

Lemma 2.2. *A tiling* $\mathcal{T}$ *is of finite local complexity, iff there are finitely many types of connected clusters containing exactly two tiles.*

Proof. If $\mathcal{T}$ is of finite local complexity there are finitely many types of r–clusters for every $r > 0$, and every connected two–tile–cluster is contained in a r-cluster for an appropriate r, so one direction is trivial.

For the other direction let k be the number of types of connected two–tile–clusters. So there are at most k possibilities to add a tile to another tile. Moreover, there are $k-1$ possibilities to add a tile to one of the tiles of a cluster consisting of more than one tile, so there are at most $k2(k-1)$ types of connected three–tile–clusters, $k2(k-1)3(k-1)$ types of connected four–tile–clusters, ..., $(n-1)!(k-1)^{n-2}$ types of connected n–tile–clusters. Since every tile has a volume larger than an appropriate $s > 0$, every r–cluster $\mathcal{C}_r(x, \mathcal{T})$, contains a finite number of tiles. So, for every $r > 0$ there is $n \in \mathbb{N}$, such that $C_r(x, \mathcal{T})$ contains at most n tiles. Therefore there are only finitely many types of r–clusters in $\mathcal{T}$. $\square$

The following result is well–known, see e.g. [Dan1, Sol1].

Theorem 2.3. *Every primitive inflation species is weakly repetitive.*
Every weakly repetitive species of finite local complexity is repetitive.

Remarks: 1. Finite local complexity, weak repetitivity and repetitivity also apply to tilings instead of species: In the above definition, change 'species $\mathbf{S}$' into 'tiling $\mathcal{T}$', 'in $\mathbf{S}$' into 'in $\mathcal{T}$' and 'for all $\mathcal{T}' \in \mathbf{S}$' into 'in $\mathcal{T}$'. Analogously, Lemma 2.2 applies to species, too.

2. One of the most remarkable properties of the tilings discovered by Berger [Ber], Penrose, Ammann, etc. was the fact, that they were nonperiodic *and* repetitive. There are many trivial examples of nonperiodic tilings (e.g. consider the canonical tiling by unit squares, and dissect one square into two rectangles), but usually the repetitive ones are of more interest, e.g., in connection with quasicrystals or pure point diffraction.

3. A weakly repetitive tiling is not necessarily of finite local complexity. There may be infinitely many types of clusters of bounded diameter. In particular, there may be infinitely many types of clusters containing exactly two tiles. In this case, most of these clusters are very rare, i.e., the corresponding $R_{\mathcal{C}}$ is very large. Danzer [Dan2] gave examples for inflation tilings, which are weakly repetitive, but not repetitive (and hence not of finite local complexity).

4. In a repetitive tiling, every cluster type with diameter less than a given $r > 0$ occurs in *every* cluster of diameter R_r. In particular, there are only finitely many types of r–clusters, so every repetitive tiling is of finite local complexity.

5. The definition 2.1 of a species corresponds to the definitions of LI–classes (LI for locally isomorphic resp. locally indistinguishable , cf. e.g. [Baa]) and tiling–spaces (cf. e.g. [Sol1] or [LMS1]). E.g., if a species $\mathbf{S}$ is primitive, for every $\mathcal{T} \in \mathbf{S}$ both the LI–class $LI(\mathcal{T})$ and the tiling space $X_{\mathcal{T}}$ are equal to $\mathbf{S}$.

An important result is the following:

Theorem 2.4 (Solomyak [Sol2]). *Let* $\mathbf{S} = \mathbf{S}(\mathcal{F}, \mathrm{infl})$ *be of finite local complexity. An inflation tiling* $\mathcal{T} \in \mathbf{S}$ *is aperiodic iff there is exactly one tiling* $\mathcal{T}' \in \mathbf{S}$ *where* $\mathrm{infl}(\mathcal{T}') = \mathcal{T}$.

We will make use of this in chapter 4.

Another very related term is self–similarity. A tiling $\mathcal{T}$ is called *self–similar*, if there exists an number $\eta > 1$ such that every tile in $\eta\mathcal{T}$ is the union of tiles in $\mathcal{T}$. Not every self–similar tiling is an inflation tiling — according to Definition 2.1 — and not every inflation tiling is self–similar. But some moments of thought yield the following:

Lemma 2.5. *Every inflation species contains a self–similar tiling.*
Every self-similar tiling $\mathcal{T}$ *of finite local complexity gives rise to an inflation species (not necessarily containing* $\mathcal{T}$ *itself).*

In this paper we stick to the well–established term species, but hold in mind the the connections with the other terms mentioned above.

Definition 2.3. *Let* $\mathcal{T}$ *be a tiling in* $\mathbb{R}^d$ *with protoset* $\mathcal{F}$. *The* $\mathbb{Z}$*–span* $\langle \mathcal{T} - \mathcal{T} \rangle_{\mathbb{Z}}$ *of*

$$\mathcal{T} - \mathcal{T} = \{ t \in \mathbb{R}^d \,|\, \exists T_i + t_1, T_i + t_2 \in \mathcal{T} : t_1 - t_2 = t \}$$

is called the translation module $\mathsf{T}(\mathcal{T})$ of $\mathcal{T}$.

If it is obvious which tiling we mean, we simply write T instead of $\mathsf{T}(\mathcal{T})$.

Theorem 2.6. *If* $\mathbf{S} = \mathbf{S}(\mathcal{F}, \mathrm{infl})$ *is a primitive inflation species, then for all* $\mathcal{T}, \mathcal{T}' \in \mathbf{S}$: $\mathsf{T}(\mathcal{T}) = \mathsf{T}(\mathcal{T}')$.

Proof. Let $\mathcal{T}, \mathcal{T}' \in \mathbf{S}$ and $T_i + t_1, T_i + t_2 \in \mathcal{T}$ ($T_i \in \mathcal{F}$). So $t_1 - t_2 \in \mathsf{T}(\mathcal{T})$. Let $\mathcal{C} \subseteq \mathcal{T}$ be a cluster containing $T_i + t_1$ and $T_i + t_2$. Because of Theorem 2.3 $\mathcal{T}'$ contains a translate of $\mathcal{C}$, say $\mathcal{C} + t$. So $T_i + t_1 + t, T_i + t_2 + t \in \mathcal{T}'$ and $t_1 + t - (t_2 + t) = t_1 - t_2 \in \mathsf{T}(\mathcal{T}')$, showing $\mathcal{T} - \mathcal{T} \subseteq \mathcal{T}' - \mathcal{T}'$. The same argument with $\mathcal{T}$ and $\mathcal{T}'$ interchanged shows $\mathcal{T}' - \mathcal{T}' \subseteq \mathcal{T} - \mathcal{T}$. The equality of these sets implies equality of their $\mathbb{Z}$–spans. □

Because of this we will speak of the translation module $\mathsf{T}(\mathbf{S})$ of a species $\mathbf{S}$, if $\mathbf{S}$ is primitive.

A lattice $L \subseteq \mathbb{R}^d$ is a discrete subgroup $(L, +)$ of $(\mathbb{R}^d, +)$. Usually it is described as the $\mathbb{Z}$–span of d linearly independent vectors $u_1, \ldots, u_d$: $L = \langle u_1, \ldots, u_d \rangle_{\mathbb{Z}} := \{ x \in \mathbb{R}^d \,|\, x = \lambda_1 u_1 + \cdots + \lambda_d u_d, \lambda_i \in \mathbb{Z} \}$. The factor group $\mathbb{R}^d / L$ is called the fundamental domain of L and can be described by $F_L = \{ x \in \mathbb{R}^d \,|\, x = \alpha_1 u_1 + \cdots + \alpha_d u_d, 0 \leq \alpha_i < 1 \}$ with addition mod L. Regarded as a point set in $\mathbb{R}^d$, the set $\mathrm{cl}(F_L) = \{ x \in \mathbb{R}^d \,|\, x = \alpha_1 u_1 + \cdots + \alpha_d u_d, 0 \leq \alpha_i \leq 1 \}$ is called *parallelepiped* in general; or, in relation with the corresponding lattice, the *fundamental parallelepiped (of L)*.

If $\mathcal{T}$ is an inflation tiling, $\mathsf{T} = \mathsf{T}(\mathcal{T})$ is a $\mathbb{Z}$-module of finite rank. The $\mathbb{R}$-span of T is $\mathbb{R}^d$. Therefore $\mathrm{rank}(\mathsf{T}) \geq d$; and $\mathrm{rank}(\mathsf{T}) = d$ is equivalent with T being a lattice. The same is true for primitive species.

Theorem 2.7. *If* $\mathbf{S}(\mathcal{F}, \mathrm{infl})$ *is a primitive inflation species with inflation factor* η *and* $\mathsf{T}(\mathbf{S})$ *is a lattice, then* $\eta^d \in \mathbb{N}$.

Proof. T contains all translations which map tiles of $\mathcal{T}$ onto tiles of the same type in $\mathcal{T}$. In particular T contains all translations which map supertiles onto supertiles of the same type as a subset. Because of the inflation this set is $\eta\mathsf{T}$. So $\eta\mathsf{T}$ is a sublattice of T with index $[\mathsf{T} : \eta\mathsf{T}] = \eta^d$. As a group index η^d is an integer. □

Because of Theorems 2.7 and 2.1 we can restrict ourselves to tilings with integer factor, whenever we consider primitive inflation species $\mathbf{S}$ where $\mathsf{T}(\mathbf{S})$ is a lattice.

Theorem 2.8. *If* $\mathcal{T}$ *is a tiling with a finite protoset* $\mathcal{F}$ *and* $\mathsf{T}(\mathcal{T})$ *is a lattice, then* $\mathcal{T}$ *is of finite local complexity.*

Proof. Let $\mathcal{T}$ be not of finite local complexity. Because of Lemma 2.2 there are infinitely many types of connected two–tile–clusters in $\mathcal{T}$. Therefore there exist two prototiles $T, T' \in \mathcal{F}$, such that $\mathcal{T}$ contains clusters of infinitely many types, say, $\{T, T' + t_i\}$ for $t_i \in \mathbb{R}^d \, (i \in \mathbb{N})$, $t_i \neq t_j$ for $i \neq j$, and $\|t_i\|$ bounded by the maximum diameter s of all prototiles.

For every $i, j \in \mathbb{N} : t_i - t_j \in \mathcal{T} - \mathcal{T} \subset \mathsf{T}(\mathcal{T})$. Therefore $\mathsf{T}(\mathcal{T}) \cap 2s\mathbb{B}^d$ contains infinitely many elements. Hence $\mathsf{T}(\mathcal{T})$ is not a lattice. □

3. From Tilings to Delone Multisets

A Delone set is a subset $V \subset \mathbb{R}^d$ which is *uniformly discrete* (i.e., there is an $r > 0$ such that every ball of radius r contains at most one $x \in V$), and which is *relatively dense* (i.e., there is an $R > 0$, such that every ball of radius R contains at least one $x \in V$). In many cases nonperiodic repetitive structures are described in terms of Delone sets or, more general, in terms of 'coloured' Delone sets, instead in terms of tilings. This leads to the following (cf. [LMS1]):

Definition 3.1. *A subset* $\boldsymbol{V} = (V_1, \ldots, V_m)^T \subset \mathbb{R}^d \times \cdots \times \mathbb{R}^d$ *is a* Delone multiset, *if each* V_i *is a Delone set, and if* $\bigcup_{i=1}^m V_i$ *is a Delone set.*

Such a set can be regarded as a Delone set, where the points have a type or a colour: V_i is the set of all points of type (or colour) i. In this sense we will use Delone multisets to describe structures in $\mathbb{R}^d$. We write $(V_1, \ldots, V_m)$ instead of $V_1 \times \cdots \times V_m$ to emphasize the fact that we consider m–tupels of elements of $\mathbb{R}^d$.

Here we don't allow a point to have more than one colour, i.e., we have always $V_i \cap V_j = \emptyset$ for $i \neq j$. This is not a restriction: Because there is a finite number m of colours, the number of possibilities of combinations of colours is $2^m - 1$, which can be directly translated into a Delone multiset with $2^m - 1$ colours, where each point has exactly one colour.

Definition 3.2. *Let* $\boldsymbol{V} = (V_1, \ldots, V_m)$ *be a Delone multiset. A set* $\mathcal{C} = (C_1, \ldots, C_n)$*, where* $C_i \subset V_i$ *and* C_i *is finite for all* $i \leq m$ *is a* cluster *in* $\boldsymbol{V}$.
The set $\mathcal{C}_r(x, \boldsymbol{V}) := \big((x + r\mathbb{B}^d) \cap V_1, \ldots, (x + r\mathbb{B}^d) \cap V_m\big)$*, where* $r > 0$*, is called* r–cluster (in x)*. Moreover,* $\mathcal{C}_0(x, \boldsymbol{V}) := \big(\{x\} \cap V_1, \ldots, \{x\} \cap V_m\big)$.

Note that any $\mathcal{C}_0(x, \boldsymbol{V})$ contains at most one point. In contrast, a zero–cluster in a tiling may contain more than one tile.

A cluster consisting of a single point x with colour i is, in the context of Delone multisets, written as $(\emptyset, \ldots, \emptyset, \{x\}, \emptyset, \ldots, \emptyset)^T$. For shortness, in this case we may write $(\{x\}, i)$.

The *support* of a cluster $\mathcal{C} = (C_1, \ldots, C_n)$ is $\mathrm{supp}(\mathcal{C}) = \bigcup_{i=1}^{n} C_i$.

As with tilings, a Delone multiset is called nonperiodic, if $\boldsymbol{V} + t := (V_1 + t, \ldots, V_m + t) = \boldsymbol{V}$ implies $t = 0$. Since defined by clusters, the terms weakly repetitive, repetitive and finite local complexity translate directly to Delone multisets, resp. families $\mathbf{S}$ of Delone multisets.

Lemma 2.2 extends to Delone multisets in the following way:

Lemma 3.1. *There is an $s > 0$ such that holds: $\boldsymbol{V}$ is of finite local complexity, iff there are only finitely many types of clusters containing exactly two elements $(\{x\}, i), (\{y\}, j)$ with $\|x - y\| \leq s$.*

Proof. The same argument as in the proof of Lemmma 2.2 applies here. A 2–point–cluster of diameter less than s can be extended to a n–point–cluster of diameter less than $(n-1)s$ in not more than $(n-1)!(k-1)^{n-2}$ ways. Since $\mathrm{supp}(\boldsymbol{V}) = \bigcup_{i=1}^{m} V_i$ is uniformly discrete, every r–cluster in $\boldsymbol{V}$ can contain at most finitely many, say, n points of $\boldsymbol{V}$. Let $s = r/(n-1)$. There are only finitely many types of n–point–clusters of diameter less than $(n-1)s = r$, and therefore only finitely many types of k–point–clusters $(1 \leq k \leq n)$ of diameter less than r. So there are only finitely many types of r–clusters. □

Similar as tilings generated by inflation, many nonperiodic Delone multisets can be generated by substitution rules. This can be done very similar to the definition of inflation species, resulting in a species of Delone multisets (cf. Fig. 5, p. 16). Or, according to [LMS2], in the following way:

Definition 3.3. *An $m \times m$–matrix $\Phi = (\Phi_{ij})_{1 \leq i,j \leq m}$, where each Φ_{ij} is a finite (possibly empty) set of mappings $x \mapsto \eta x + a_{ijk}$, where $\eta > 1$, $1 \leq i, j \leq m$, $1 \leq k \leq \#(\Phi_{ij})$, $a_{ijk} \in \mathbb{R}^d$, is called* matrix function system*.*
A Delone multiset $\boldsymbol{V} = (V_1, \ldots, V_m)$ is called substitution Delone multiset *if there is a matrix function system Φ, such that $\Phi(\boldsymbol{V}) = \boldsymbol{V}$.*
A substitution Delone multiset is called a lattice substitution system*, if $V_1 \cup \cdots \cup V_m$ is a lattice.*

Because of the obvious relation between the inflation factor of tilings and the role of η in the above definition, we will refer to this η in the following also as inflation factor.

With this definition, the substitution is realized by matrix operations. The substitution of $\mathcal{C} = (C_1, C_2, \ldots, C_m)^T$ is

$$\Phi\ \mathcal{C} = \left(\bigcup_{j=1}^{m} \bigcup_{\varphi \in \Phi_{1j}} \varphi(C_j), \ldots, \bigcup_{j=1}^{m} \bigcup_{\varphi \in \Phi_{mj}} \varphi(C_j) \right)^T .$$

In a similar way one can compute Φ^k, which gives the k–times applied substitution.

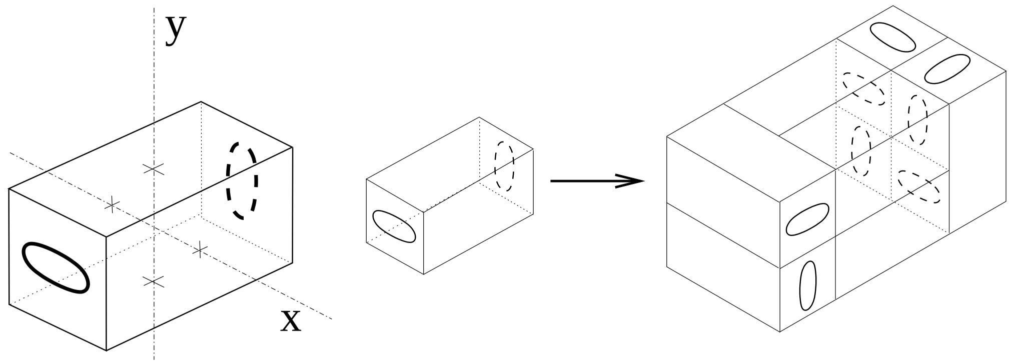

FIGURE 2. A prototile (left) and the inflation rule for a 3–dimensional inflation species $\mathbf{S}$ (right), where $\mathsf{T}(\mathbf{S})$ is a lattice

Two simple examples of lattice substitution systems are given in chapter 5 (Examples 1 and 2).

An important point is, that tilings can be described in terms of Delone multisets, and vice versa. Consider the three-dimensional species, which inflation is described in Fig. 2. The prototiles are rectangular parallelepipeds, with side lengths 1,1 and 2, which are decorated with two elliptical decorations on the two square faces, as shown. One prototile, say, T_1, is shown on the left part of Fig. 2. The other prototiles we get by rotating T_1 about the axis x (resp. y) through $\pi/2, \pi, 3\pi/2$ (resp. $\pi/2, 3\pi/2$). (Notice that the rotation about x through π fixes the tile, but alters the decoration; therefore these are regarded as different prototiles.)

From each of these tilings we can deduce a Delone multiset in a canonical fashion (cf. Fig. 3): Each tile can be dissected into two cubes. Take the centres of these cubes, and define their type corresponding to which of the six prototiles they belong to, and to which one of the two cubes they belong to. This gives 12 types (or colours) for the Delone multiset $\boldsymbol{V} = (V_1, \ldots, V_{12})^T$, and the support $\mathrm{supp}(\boldsymbol{V}) = \bigcup_{i=1}^{12} V_i$ is a translate of $\mathbb{Z}^3$.

In this example it is obvious, that the relation between tilings and Delone multisets is one–to–one: From a tiling of the considered species we get a Delone multiset in a unique way, and vice versa such a Delone multiset determines a unique tiling of the species. This property is called 'mutual local derivability' (cf. [BSJ]). In the following Definition this will be made precise.

For the following Definition and Theorem let A, B be two tilings in $\mathbb{R}^d$, or two Delone multisets in $\mathbb{R}^d \times \cdots \times \mathbb{R}^d$, or one Delone multiset in $\mathbb{R}^d \times \cdots \times \mathbb{R}^d$ and one tiling in $\mathbb{R}^d$.

Definition 3.4. *A is* locally derivable (with radius r) *from B, if for all x, y in $\mathbb{R}^d$ holds:*

$$\mathcal{C}_r(x, A) = \mathcal{C}_r(y, A) + (x - y) \Rightarrow \mathcal{C}_0(x, B) = \mathcal{C}_0(y, B) + (x - y).$$

If A is locally derivable from B and B is locally derivable from A, then A and B are mutual locally derivable.

It is not hard to see that mutual local derivability is an equivalence relation.

Theorem 3.2. *Let A and B be mutual locally derivable. A is of finite local complexity, iff so is B.*

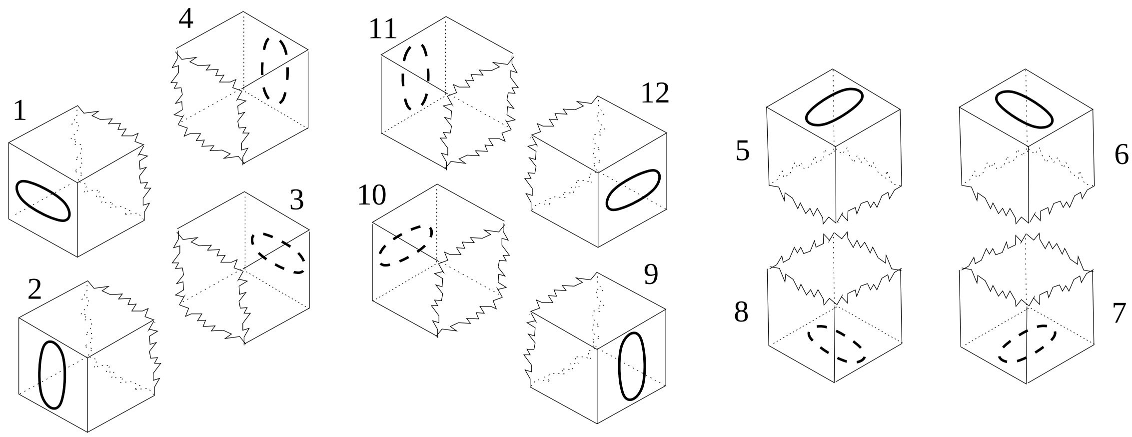

FIGURE 3. Deduction of the point types of the Delone multisets corresponding to the tilings of Fig. 2

Proof. Let A be of finite local complexity and $r > 0$ as in Definition 3.4. For every $s > 0$ there are finitely many types of $(r+s)$–clusters in A. Of course, A is the union of all $(r+s)$–clusters of A.

$$\begin{aligned}
\mathcal{C}_{r+s}(x, A) &= \mathcal{C}_{r+s}(y, A) + (x - y) \\
\Rightarrow \forall x' \in (s\mathbb{B}^d + x), y' \in (s\mathbb{B}^d + y) : \mathcal{C}_r(x, A) &= \mathcal{C}_r(y, A) + (x - y) \\
\Rightarrow \forall x' \in (s\mathbb{B}^d + x), y' \in (s\mathbb{B}^d + y) : \mathcal{C}_0(x, B) &= \mathcal{C}_0(y, B) + (x - y) \\
\Rightarrow \mathcal{C}_s(x, B) &= \mathcal{C}_s(y, B) + (x - y)
\end{aligned}$$

So an $(r+s)$–cluster in A at x determines the type of the corresponding s–cluster in B at x. In particular, finitely many types of $(r+s)$–clusters in A force finitely many types of s–clusters in B. □

In [LMS2] is shown, that for every tiling $\mathcal{T}$ there is a Delone multiset $\boldsymbol{V}_{\mathcal{T}}$ such that $\mathcal{T}$ and $\boldsymbol{V}_{\mathcal{T}}$ are mutual locally derivable. The construction is a general way of what we did in the example in Fig. 3. In particular, if $\mathcal{T}$ is a primitive inflation tiling, $\boldsymbol{V}_{\mathcal{T}}$ can be choosen as a primitive substitution Delone multiset. This extends to species and families of Delone multisets.

It is not clear, that the construction mentioned above gives a lattice substitution system which maps points to blocks of $q \times q \times \cdots \times q$ points. But this is what we need for the following.

Theorem 3.3. *Every primitive inflation species* $\mathbf{S}$ *with factor* q*, where* $\mathsf{T}(\mathbf{S}) = \langle u_1, \ldots, u_d \rangle_{\mathbb{Z}}$ *is a lattice, contains a tiling* $\mathcal{T}$*, which is mutual locally derivable to a primitive lattice substitution system* $(\mathbf{V}, \Phi)$ *on* $\mathsf{T}(\mathbf{S})$*. Moreover,* $(\mathbf{V}, \Phi)$ *can be chosen such that the union of the entries of any column of the matrix function system* Φ *contains the* q^d *different elements of* M*, where*

$$M := \{qx + a_{i_1 i_2 \ldots i_d},\, |\, a_{i_1 i_2 \ldots i_d} = \sum_{j=1}^{d} i_j u_j,\, (i_1, \ldots, i_d) \in \{0, 1, \ldots, q-1\}^d\}$$

The latter means that Φ maps every lattice point x to the q^d lattice points lying inside the parallelepiped $q(F+x)$, where F is a fundamental parallelepiped of the lattice $\mathsf{T}(\mathbf{S})$.

Proof. We give an explicit construction of a Delone multiset with the properties above, starting with an arbitrary species $\mathbf{S} = \mathbf{S}(\mathcal{F}, \mathrm{infl})$.

Let $\mathcal{T} \in \mathbf{S}$, where $\mathbf{S}$ is primitive and $\mathsf{T}(\mathbf{S})$ is a lattice. Choose a point $x \in T_1 \in \mathcal{F}$ and consider $x + \mathsf{T}(\mathbf{S})$. Note, that if a point $z \in x + \mathsf{T}(\mathbf{S})$ is contained in a tile of type $T_i \in \mathcal{F}$, say, $T_i + t$, then, because of the role of $\mathsf{T}(\mathbf{S})$, z is in the relatively same position in $T_i + t$ as x in T_i, i.e., $z = x + t$.

For every $z \in x + \mathsf{T}(\mathbf{S})$ consider the R–cluster $\mathcal{C}_R(z, \mathcal{T})$. By Theorem 2.8 there are finitely many, say, m, types of such. According to the type i of $\mathcal{C}_R(z, \mathcal{T})$, z gets label i $(1 \le i \le m)$. Then $\mathbf{V} = (V_1, \ldots, V_m)$, where V_i is the set of all $z \in x + \mathsf{T}(\mathbf{S})$ with label i, is the Delone multiset wanted.

Now decorate $\mathrm{infl}(\mathcal{T})$ in the same way, i.e., choose a tile $T_1 + t \in \mathrm{infl}(\mathcal{T})$, consider $x + t + \mathsf{T}(\mathbf{S})$ and label the points in the same way as above, depending on the type of their R–clusters.

To determine Φ, choose $z \in x + \mathsf{T}(\mathbf{S})$ with label j (regarded as a point in $\mathcal{T}$) and take a look at the point qz (regarded as a point in $\mathrm{infl}(\mathcal{T})$, where q is the inflation factor). Of course qz lies in some translate of $qF + v$, $v \in t + q(x + \mathsf{T}(\mathbf{S}))$ (where F is the fundamental parallelepiped of $\mathsf{T}(\mathbf{S})$). The positions of the q^d lattice points $z_1, z_2, \ldots, z_{q^d}$ in $qF + v$ give the translation vectors $a_1, a_2, \ldots, a_{q^d}$ via $a_k := z_k - \eta z$. The labels of the points z_k gives the labels of a_k. Now let $\Phi_{ij} := \{\varphi \,|\, \varphi(x) = qx + a_k,\, a_k \,\text{has label}\, i\}$, thus giving the matrix function system $\Phi = (\Phi_{ij})$.

Since the images of all R–clusters of the same type in $\mathcal{T}$ are of the same type in $\mathrm{infl}(T)$, Φ maps points of the same type onto clusters of the same type. Therefore Φ is well defined.

The mutual local derivability of $\mathcal{T}$ and $\mathbf{V}$ is clear by the construction. So the last point to fulfil is the condition $\Phi(\mathbf{V}) = \mathbf{V}$. According to Lemma 2.5 every primitive inflation species $\mathbf{S}(\mathcal{F}, \mathrm{infl})$ contains a tiling $\mathcal{T}'$ where $\mathrm{infl}^k(\mathcal{T}') = \mathcal{T}'$ for some appropriate $k \in \mathbb{N}$. Above we considered an arbitrary tiling $\mathcal{T} \in \mathbf{S}$, so we apply the construction to $\mathcal{T}'$ in particular, getting $(\mathbf{V}, \Phi)$ where $\Phi(\mathbf{V}) = \mathbf{V}$. Maybe we have to replace infl by infl^k, but by Theorem 2.1 this does not alter the considered species. □

Now we have, that every *superelement*, i.e., every cluster $\Phi^k((\{x\}, i))$ is of the same shape, namely $\mathrm{supp}\big(\Phi^k((\{x\}, i))\big)$ is a block of $q^k \times q^k \times \cdots \times q^k$ points. Therefore it will be useful to compare two superelements by checking if the points at the same relative positions are of the same type.

Definition 3.5. *Let* $\mathrm{supp}\big(\Phi^k(\{x\}, i)\big) = t + \mathrm{supp}\big(\Phi^k(\{y\}, j)\big)$. *Then*

$$\Phi^k(\{x\}, i) \sqcap \Phi^k(\{y\}, j) := \{z \in \mathbb{R}^d \,|\, \exists \ell : (\{z\}, \ell) \in \Phi^k(\{x\}, i),\ (\{z + t\}, \ell) \in \Phi^k(\{y\}, j)\}$$

In plain words: $\big(\Phi^k(\{x\}, i)\big) \sqcap \big(\Phi^k(\{y\}, j)\big)$ gives the set of points, on which the two superelements coincide. Or, equivalently, it gives the set of points, where the two superelements have the same colour.

4. Does $\boldsymbol{V}$ consist of Model Sets?

The context in [LMS2] is slightly more general than in this text. Here we consider affine maps of the form $\varphi(x) = qx + a$. (For consistence with [LM] and [LMS2], and since we deal only with integer factors in this chapter, we will use q instead of η to denote the inflation factor). The linear part can be written as $x \mapsto Qx$, where $Q := qI$ (I the identity matrix). In [LMS2] the authors consider affine maps where the linear part is *any* expansive linear map $x \mapsto Qx$. i.e., there is a $c > 1$ such that $d(Qx, Qy) \geq c\, d(x, y)$ for all $x, y \in \mathbb{R}^d$ (d the Euclidean distance).

Since we started with tilings, our restriction to maps of the first form is not too strict. Even if we would allow for tilings this more general expansive map — instead of multiplication by η — some possibilities will be ruled out:

E.g., if Q contains a rotation σ, there must be a $k \in \mathbb{N}$ such that $\sigma^k = \mathrm{id}$. Otherwise there would be infinitely many tile types up to translation. By Theorem 2.1 we can go over to the species $\mathbf{S}(\mathcal{F}, \mathrm{infl}^k)$, getting an equivalent lattice substitution system[1], where Q does not contain a rotation. .

In a similar way we can rule out the occurance of reflections or shears.

Moreover, in our setting we need not care about 'legality' of clusters, since we did start with primitive inflation tilings of finite local complexity, therefore repetitive tilings. In such tilings, every cluster is legal.

In the earlier article [LM], the authors gave conditions to decide, if the Delone multiset did consist of model sets of a special form (namely, where the internal space G is based on the Q–adic completion of L). This did not cover examples like

$$a \to aba,\ b \to bab$$

giving sequences $\ldots abababab \ldots$. These are, regarded as periodic sequences on $\mathbb{Z}$, indeed model sets, but not related to a Q–adic (here: 3–adic) completion of $L = \mathbb{Z}$. So with the tools of [LM] this example could not be handled. In [LMS2] the conditions determine, whether the Delone multiset did consist of model sets of *any* form. This makes things more technical.

Definition 4.1 (cf. [LMS2])**.** *Let $(\Phi, \mathbf{V})$ be a lattice substitution system and $L = supp(\mathbf{V})$. Let $L' := L_1 + L_2 + \cdots + L_m$, where $L_i := \langle V_i - V_i \rangle_{\mathbb{Z}}$.*
For $a \in L$, let

$$\Phi_{ij}[a] := \{\varphi \in \Phi_{ij} \,|\, \varphi(y) = a \mod qL', V_j \subseteq y + L'\}$$

Furthermore, let

$$\Phi[a] := \bigcup_{1 \leq i,j \leq m} \Phi_{ij}[a]$$

$(\mathbf{V}, \Phi)$ *admits a* modular coincidence relative to qL', *if there is $k \in \mathbb{N}, a \in \mathbb{R}^d$ such that all elements $\varphi \in (\Phi^k)_{ij}$, where $\varphi(x) = qx + a$, are contained entirely in one row of Φ^k.*

Remark: In [LMS2] it is stated, that $(\mathbf{V}, \Phi)$ admits a modular coincidence, if and only if some V_i contains a translate of some lattice L'. Moreover, from Theorem 4 of [LM] follows,

[1] This is true in our context, but it need not to be true for tilings with finitely many tile types up to isometry (but not up to translations), like the pinwheel tilings (cf. [Rad]).

if such a lattice exists, every V_i is a union of translates of lattices (up to a set N of relative density 0):

$$V_i = \bigcup_{k\in\mathbb{N}} (q^k L + t_k) \cup N \tag{1}$$

This facts we will use later.

Theorem 4.1 (Lee, Moody, Solomyak)**.** *Let $(\boldsymbol{V},\Phi)$ be a lattice substitution system, $\boldsymbol{V} = (V_1,\ldots,V_m)$ mutual locally derivable with a primitive inflation tiling. Let $L' = L_1 + \cdots + L_m$, where $L_i := \langle V_i - V_i\rangle_{\mathbb{Z}}$. Then the following are equivalent.*

1. *Each V_i is a regular model set.*
2. *A modular coincidence relative to $q^k L'$ occurs in Φ^k.*
3. *$\mathbf{V}$ is pure point diffractive.*

We should mention, that in many 'well–behaving' cases this reduces to one of the following:

There is a $k \in \mathbb{N}$, such that in one row of Φ^k all entries have a common element. Or:

There is $a \in \mathbb{R}^d, k \in \mathbb{N}$, such that all maps occuring in Φ of the form $\varphi(x) = q^k x + a$ are contained entirely in one row of Φ^k.

So this theorem gives in principle an algorithm to decide, if a given lattice substitution system does consist of model sets or not: Just compute powers of Φ and check for coincidences. In [LM] the authors used this algorithm to show, that the sphinx tiling and the d–dimensional chair tiling arose from model sets.

If a tiling does not arise from a model set, this algorithm never terminates. So it is useful to have conditions giving a negative answer in finite time for a given tiling or lattice substitution system.

Theorem 4.2. *Let $(\mathbf{V},\Phi)$ be a primitive nonperiodic lattice substitution system with inflation factor $q \in \mathbb{N}$ on a lattice $L \subseteq \mathbb{R}^d$. If, for every i, the Φ_{ij} are pairwise disjoint $(1 \le j \le m)$, then no V_k contains a translate of a d–dimensional lattice.*

Proof. Suppose there is a V_i which contains a translate $L' + t$ of a d–dimensional lattice L'. We can restrict ourselves to $t = 0$.

With $L = \langle u_1,\ldots,u_d\rangle_{\mathbb{Z}}$ we define the bijection

$$\pi : L \to \mathbb{Z}^d,\ \pi(\lambda_1 u_1 + \cdots + \lambda_d u_d) = (\lambda_1,\ldots,\lambda_d)^T \tag{2}$$

There is a lattice $L' \subseteq V_i$ iff there is a lattice $\pi^{-1}(L') \subseteq \pi^{-1}(V_i) \subseteq \mathbb{Z}^d$. Therefore we can restrict ourselves to $L = \mathbb{Z}^d$.

Next we show that under our assumption there is also a lattice in $\pi(V_i)$ with axis–parallel basis vectors, i.e., with basis vectors $\mu_1 e^{(1)},\ldots,\mu_d e^{(d)}$, where $e^{(i)}$ denotes the canonical basis vectors, and $\mu_i \in \mathbb{Z}$.

The generator matrix M of $\pi(L')$, i.e., the matrix with columns $u_1,\ldots,u_d$, is regular and has integer entries. Therefore $M^{-1} \in \mathbb{Q}^{d\times d}$. Let $v^{(i)} := M^{-1}e^{(i)}$. Because $v^{(i)} \in \mathbb{Q}^d$ there are integers $\mu_1,\ldots,\mu_n$ such that $\mu_i v^{(i)} \in \mathbb{Z}^d$. Furthermore $w^{(i)} := \mu_i e^{(i)} = M\mu_i v^{(i)} \in \pi(L')$ (since every lattice vector is of the form Mv, where $v \in \mathbb{Z}^d$) and $\tilde{L} = \langle w^{(1)},\ldots,w^{(d)}\rangle_{\mathbb{Z}}$ is the lattice wanted.

If there is no such lattice $\tilde{L}$, there can't be a lattice $L' \subseteq V_i$.

For every i, the Φ_{ij} are pairwise disjoint. With the notation of Def. 3.5, this reads

$$\forall\, 1 \le i < j \le m : \Phi((\{0\}, i)) \sqcap \Phi((\{0\}, j)) = \emptyset$$

and by induction follows

$$\forall\, k \in \mathbb{N} : \forall\, 1 \le i < j \le m : \Phi^k((\{0\}, i)) \sqcap \Phi^k((\{0\}, j)) = \emptyset$$

So we have for two superlements of k–th order, say, A and B, with $\mathrm{supp}(A) + t = \mathrm{supp}(B)$:

$$(A + t) \sqcap B \ne \emptyset \Rightarrow A \text{ and } B \text{ are of the same type.} \tag{3}$$

In plain words: If we know the type of a single point $z \in \boldsymbol{V}$ resp. $z \in \pi(\boldsymbol{V})$ and its relative position in the k–th order superelement it belongs to, so we know the type of the superelement.

Any given nonperiodic $\boldsymbol{V}$ is — as a Delone multiset on a lattice — of finite local complexity. So, by Theorem 2.4, there is a unique Delone multiset $\boldsymbol{V'}$ with $\Phi(\boldsymbol{V'}) = \boldsymbol{V}$. Therefore the positions of all superelements in $\boldsymbol{V}$ are known. The type of a single $z \in \boldsymbol{V}$ — resp.[2] $z \in \pi(\boldsymbol{V})$ — therefore determines the type of every superelement it belongs to.

Now we deduce from the d–periodicity of the lattice $\tilde{L}$ the d–periodicity of $\boldsymbol{V}$.

All points in $\tilde{L}$ are of the same type (resp. the same colour). I.e., if $z \in \tilde{L}$, then every $z + \sum_{i=1}^d \lambda_i w^{(i)}$ is of the same type as z. If some $z + \sum_{i=1}^d \lambda_i w^{(i)}$ is in the same relative position in a superelement $A + t$ of k–th order as z is in a superelement $B + s$ of k–th order, it follows $A = B$.

By Theorem 3.3 the superelements are blocks of $q^k \times \cdots \times q^k$ points. So we can describe the relative position in a superelement 'modulo q^k'. Two points $z = (z_1, \ldots, z_d)^T, z' = (z'_1, \ldots, z'_d)^T \in \pi(L')$ — or, more general, in $\pi(L) = \mathbb{Z}^d$ — are in the same relative position in their k–th order superelement iff

$$\forall 1 \le i \le d : \; z_i \equiv z'_i \mod q^k$$

Let $w_{max} := \max\{\|w^{(i)}\| \,|\, 1 \le i \le d\}$. Choose $k \in \mathbb{N}$ such that $q^k \ge w_{max}$. So every superelement of k–th order now contains at least one element $z \in \tilde{L}$, and its type is determined by the relative position of z.

Let now $z^{(0)} = (z_1^{(0)}, \ldots, z_d^{(0)})^T \in \tilde{L}$, and $A^{(0)} + t_0$ the k–th order superelement containing $z^{(0)}$. For every point $z' = (z'_1, \ldots, z'_d)^T \in z^{(0)} + q^k \tilde{L}$ holds

$$\forall 1 \le i \le d : \; z'_i = z_i^{(0)} + \lambda_i q^k \|w^{(i)}\| \equiv z_i^{(0)} \mod q^k \; (\lambda_i \in \mathbb{Z})$$

Therefore all k–th order superelements containing an element of $z^{(0)} + q^k \tilde{L}$ are of the same type $A^{(0)}$. So the set $A^{(0)} + t_0 + q^k \tilde{L}$ is a d–periodic subset of $\pi(\boldsymbol{V})$ of positive relative density. (To be precise: $\mathrm{dens}(A^{(0)} + t_0 + q^k \tilde{L}) / \mathrm{dens}(\pi(\boldsymbol{V})) = (\prod_{i=1}^d \|w_i\|)^{-1}$.)

If this set covers every superelement of k–th order, we are done. Otherwise there are superelements of k–th order which are not covered until now. In this case, we repeat this construction with another element $z^{(1)} \in \tilde{L}$, $z^{(1)} \notin A^{(0)} + t_0 + q^k \tilde{L}$. This gives a d–periodic set $A^{(1)} + t_1 + q^k \tilde{L}$ of k–th order supertiles, which is disjoint with $A^{(0)} + t_0 + q^k \tilde{L}$ and of the *same* positive relative density.

[2]Here and in what follows $\pi(\boldsymbol{V})$ is to be read as the Delone multiset $(\pi(V_1), \ldots, \pi(V_m))^T$ on $\mathbb{Z}^d$

We repeat this construction (to be precise $N = \prod_{i=1}^{d} \|w_i\|$ times), always beginning with an element $z^{(i)}$ not contained in one of the former sets $A^{(j)} + t_j + q^k \tilde{L}$, $0 \leq j < i$. Every element of $\pi(\boldsymbol{V})$ is contained in some k–th order supertile, and therefore we get the whole Delone multiset this way:

$$\begin{aligned} \pi(\boldsymbol{V}) &= \bigcup_{0 \leq i \leq N} (A^{(i)} + t_i + q^k \tilde{L}) \\ &= (\bigcup_{0 \leq i \leq N} A^{(i)} + t_i) + q^k \tilde{L} \end{aligned}$$

It follows that $\pi(\boldsymbol{V})$ is d-periodic and therefore $\boldsymbol{V}$, which is a contradiction. □

Applying Theorem 4.1 to this, we get the following result.

Corollary 4.3. *Let $(\mathbf{V}, \Phi)$ fulfil the conditions of Theorem 4.2. Then $\mathbf{V}$ is not pure point diffractive and does not consist of model sets.*

A more general result, but also more technical and more difficult to apply, is the following.

Theorem 4.4. *Let $(\boldsymbol{V}, \Phi)$ be a primitive lattice substitution system with inflation factor $q \in \mathbb{N}$ on a lattice $L = \langle u_1, \ldots, u_d \rangle_{\mathbb{Z}} \subseteq \mathbb{R}^d$. Let $\boldsymbol{V} = (V_1, \ldots, V_m)$ contain an one–dimensional infinite sequence S of equidistant points of the form*

$$\cdots xaxbxax \cdots xaxbxax \cdots xaxbxax \cdots$$

with the following properties:

1. *x stands for $r-1$ arbitrary points*
2. *The convex hull of S is parallel to a lattice basis vector u_i*
3. *on $\ell - 1$ points of type a follows one point of type b*
4. $\Phi((\{0\}, a)) \sqcap \Phi((\{0\}, b)) = \emptyset$[3]
5. *In $\Phi((\{0\}, a))$ occurs an a, in $\Phi((\{0\}, b))$ occurs an b, at the same position.*

Then there is a subset of $\mathbf{V}$ with positive relative density, which contains no point of a lattice $L \subseteq V_i$, such that the period of L along the direction u_i is of length $q^k nr\|u_i\|$, where $n \in \mathbb{N} \setminus \ell\mathbb{N}, k \in \mathbb{N} \cup \{0\}$.

How to use this Theorem to show, that a lattice substitution system does not consist of model sets? First, find a sequence which fulfils the conditions of the Theorem. Second, show by an ad–hoc–method that every possible one–coloured lattice in $\mathbf{V}$ must have a period of the form in the Theorem. Third, by (1) and the remark there, if $\mathbf{V}$ consists of model sets, $\mathbf{V}$ must be the union of such lattices, up to a subset of zero density. This gives a contradiction, so $\mathbf{V}$ cannot be a model set. For an explicit example, see Section 5, Example 3.

Proof. Let V_i contain a translate $L' + t$ of a d–dimensional lattice L'. Again we can restrict ourselves to $t = 0$ and $L = \mathbb{Z}^d$, with π as in (2).

Now, with $\pi(L')$ a sublattice of $\mathbb{Z}^d$, there is also — using the same argument as in the proof of Theorem 4.2 — a sublattice $\tilde{L}$ with axis–parallel basis vectors. Since $\pi(u_i) = e^{(i)}$, the

[3]For the definition of $\sqcap$, see Def. 3.5

sequence S is parallel to one of these, say, S is parallel to $e^{(1)}$. Let $w_1 = \nu e^{(1)}$, $\nu \in \mathbb{N}$, be the corresponding basis vector of $\tilde{L}$.

We will show that $qnre^{(1)}$ $(k \geq 0, n \in \mathbb{N}/\ell\mathbb{N})$ is not a period of $\tilde{L}$. In other words, $\tilde{L}$ doesn't have a period of length qnr $(k \geq 0, n \in \mathbb{N}/\ell\mathbb{N})$ in direction $e^{(1)}$.

Because of the self–similarity, $\boldsymbol{V}$ also contains the infinite sequence $\Phi(S)$.

Let $z \in \tilde{L}$ be contained in a superelement $\Phi((\{x\}, b))$ of $\Phi(S)$. Such a z have to exist, since, if there is a lattice in V_i, *every* V_j consists of lattices (up to a set of relative density 0, cf. (1) and the preceding remark).

Now, every $z + \lambda q w_1 \in \tilde{L}$ $(\lambda \in \mathbb{Z})$, hence is of the same type (the same colour) as z. Moreover, every $z + \lambda q w_1$ $(\lambda \in \mathbb{Z})$ lies at the same relative position in its supertile.

Also, every $z + \lambda q r w_1$ $(\lambda \in \mathbb{Z})$ lies — because of the role of r — in a superelement of type $\Phi((\{x\}, a))$ or $\Phi((\{x\}, b))$, at the same relative position as z does.

$\Phi((\{0\}, a)) \sqcap \Phi((\{0\}, b)) = \emptyset$ rules out the possibility $z + \lambda q r w_1 \in \Phi((\{x\}, a))$. So every $z + \lambda q r w_1$ $(\lambda \in \mathbb{Z})$ lies in a superelement of type $\Phi((\{x\}, b))$.

On the other hand: if ℓq does not divide $\lambda q \|w_1\|$, then $z + \lambda q r w_1$ lies — because of the form of S resp. $\Phi(S)$ and the role of ℓ — in a superelement of type $\Phi((\{0\}, a))$. So we have a contradiction, considering the following points:

If $\|w_1\|$ is divisible by ℓ, the length of the period of $\tilde{L}$ in direction $e^{(1)}$ is a multiple of $q\ell r$. This case we do not want to (and cannot) rule out.

If $\|w_1\|$ is not divisible by ℓ, we can always choose λ in a way such that $\lambda q \|w_1\|$ is not divided by ℓq (e.g., $\lambda = \ell - 1$).

Now we know, that the sequence $\Phi(S)$ does not contain points of the assumed lattice $\tilde{L}$. Since $\mathbf{V}$ is repetitive, finite subsequences of $\Phi(S)$ occur in $\mathbf{V}$ with positive relative density. Moreover, for any $R > 0$, finite subsequences of $\Phi(S)$ with length larger than R occur with positive relative density. With R large enough, by the arguments above, none of these subsequences can contain points of the assumed lattice $\tilde{L}$. Altogether, we have a subset of $\mathbf{V}$ with positive relative density and containing no lattice points of $\tilde{L}$.

Now we extend this fact to period lengths of the form $q^k nr$. This is fastly done by using condition (5), which we did not need up to this point.

Since $\mathbf{V}$ is self–similar, there are also sequences

$$\cdots x\Phi^k(a)x\Phi^k(b)x\Phi^k(a)x \cdots x\Phi^k(a)x\Phi^k(b)x\Phi^k(a)x \cdots x\Phi^k(a)x\Phi^k(b)x\Phi^k(a)x \cdots$$

in $\mathbf{V}$. In these we find, because of condition (5), a sequence of the form

$$\cdots xaxbxax \cdots xaxbxax \cdots xaxbxax \cdots$$

where ℓ is the same; r is replaced by a new value r', where $r' = q^k(r-1) + q^k - 1 = q^k r$. Of course, q is the same, $\|u_i\|$ is the same, so the claim follows: There is a subset of $\mathbf{V}$ with positive relative density, containing no lattice points of a lattice with period $q^k nre^{(1)}$. $\square$

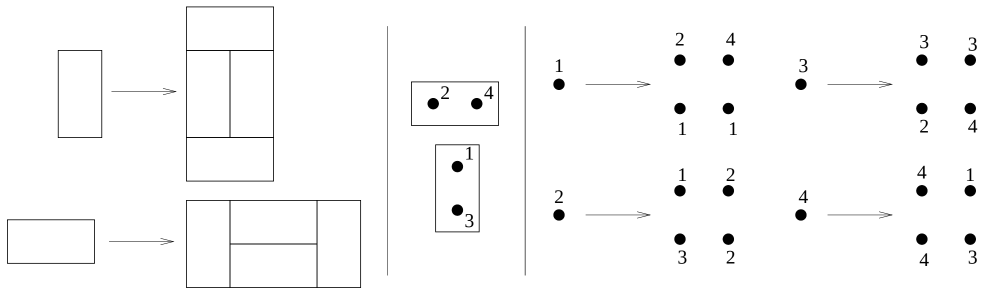

FIGURE 4. Inflation rule for the table tiling, point decoration for constructing the relating Delone multiset, inflation rule for the points (from left to right).

5. Examples and Concluding Remarks

Examples: 1. A Thue–Morse sequence can be generated by the substitution $0 \to 01$, $1 \to 10$. The matrix function system of the lattice substitution system of a Thue–Morse sequence is

$$\Phi = \begin{pmatrix} \{h_0\} & \{h_1\} \\ \{h_1\} & \{h_0\} \end{pmatrix}, \quad h_i(x) := 2x + i \quad (i \in \{0,1\})$$

Here $\Phi_{11} \cap \Phi_{12} = \emptyset, \Phi_{21} \cap \Phi_{22} = \emptyset$. Applying Theorem 4.2 shows that the Thue–Morse sequence does not consist of model sets, and is not pure point diffractive.

2. A table (or dimer or domino) tiling is shown in Fig. 5. The matrix function system for an appropriate Delone multiset mutual locally derivable with the table tiling is

$$\Phi = \begin{pmatrix} \{h_0,h_1\} & \{h_3\} & \emptyset & \{h_2\} \\ \{h_3\} & \{h_1,h_2\} & \{h_0\} & \emptyset \\ \emptyset & \{h_0\} & \{h_2,h_3\} & \{h_1\} \\ \{h_2\} & \emptyset & \{h_1\} & \{h_0,h_3\} \end{pmatrix}$$

where $h_0(x) := 2x$, $h_1(x) := 2x + (1,0)^T$, $h_1(x) := 2x + (1,1)^T$, $h_1(x) := 2x + (0,1)^T$. In every row the entries are pairwise disjoint, so again by Theorem 4.2 the table tiling can't be described by model sets, and is not pure point diffractive.

3. The lattice substitution system $(\mathbf{V}, \Phi)$ arising from the tiling shown in Figs. 2 and 3 does not consist of model sets. This cannot be shown by Theorem 4.2 but by Theorem 4.4.

To keep it short: Without loss of generality let $\text{supp}(\mathbf{V}) = \mathbb{Z}^3$. First we give a periodic one–dimensional sequence in $\mathbf{V}$. Consider $\Phi((\{0\},1))$ and $\Phi((\{0\},2))$. The support of both of them consists of eight points which lie on the vertices of a cube, or equivalently, two layers of four points. The action of Φ therefore can be described in principle by the following diagram:

$$1 \to \begin{array}{cc} 2 & 1 \\ 11 & 12 \end{array} \Big| \begin{array}{cc} 1 & 2 \\ 10 & 9 \end{array}, \qquad 2 \to \begin{array}{cc} 2 & 1 \\ 5 & 6 \end{array} \Big| \begin{array}{cc} 1 & 2 \\ 8 & 7 \end{array}$$

We are only interested in one of these two layers. Reduced to this – and forgetting about the other layer — Φ acts in the following way:

$$1 \to \begin{array}{cc} 2 & 1 \\ 11 & 12 \end{array}, \qquad 2 \to \begin{array}{cc} 2 & 1 \\ 5 & 6 \end{array}$$

Applying Φ iteratively to this patterns of 4 points, we will get patterns of $16, 64, \ldots, 4^k \ldots$ points. Now consider the top line of this pattern: It is always of the form 2 1 2 1 2 1 2 .... Therefore the line below it is of the form 11 12 5 6 11 12 5 6 11 12.... This sequence will do.

The action of Φ on 5 resp. 11 is shown in the following diagram:

$$5 \to \begin{array}{cc} 4 & 3 \\ 1 & 2 \end{array} \Big| \begin{array}{cc} 5 & 6 \\ 6 & \underline{5} \end{array}, \qquad 11 \to \begin{array}{cc} 6 & 11 \\ 5 & 10 \end{array} \Big| \begin{array}{cc} 7 & 10 \\ 8 & \underline{11} \end{array}$$

We read off that $\Phi((\{0\},5)) \sqcap \Phi((\{0\},11)) = \emptyset$, and in $\Phi((\{0\},5))$ a 5 occurs at the same relative position as 11 in $\Phi((\{0\},11))$. With notations as in Theorem 4.4 we have the symbols $a = 5, b = 11$ (and $x = 12$ or $x = 6$ in an alternating fashion) and numbers $r = 2, \ell = 2$, and the inflation factor is $q = 2$.

Therefore there is a subset of $\mathbf{V}$ of positive density, that does not contain a lattice, whose period in the considered direction is of length $q^k nr = 2^k n2\,(k \geq 0, n \in \mathbb{N} \setminus 2\mathbb{N})$. This means, it does not contain a lattice, whose period length in this direction is an even number.

Now a 'checkerboard–argument' applies. $\mathbf{V}$ partitions into 2 classes of types: Points of type $1, 3, 5, 7, 9$ and 11 are on 'white' positions, and points of type $2, 4, 6, 8, 10$ and 12 are on 'black' positions. If there is some lattice contained in some V_i, the length of its period — parallel to a canonical base vector — therefore must be an even number. From (1) we know, that the union of these lattices of even period gives a set of relative density 1. This is a contradiction. So no V_i contains a lattice, and therefore no V_i is a model set.

Remarks: 1. Many of the results collected in chapter 2 are true for a broader context than primitive inflation species $\mathbf{S}$, where $\mathcal{T}(\mathbf{S})$ is a lattice. Theorem 2.1 and Lemma 2.5 are true for any inflation species, and Lemma 2.2 and Theorem 2.8 apply also to tilings, which are not inflation tilings.

2. The first condition, Theorem 4.2, is clearly not a necessary condition for showing, that a given tiling or Delone multi set cannot be described as a model set. The second condition, Theorem 4.4, seems very artificial. Amazingly it can be applied to a lot of examples. Anyway, it also seems not to be a necessary condition. Consider the example in Fig. 2. The author neither did succeed in applying Theorem 4.1 to the corresponding lattice substitution system, nor did succeed in applying Theorem 4.4 to it.

3. There may be limits on the power of Φ, beyond which one do not have to check for coincidence. In other words: Is there a $k \in \mathbb{N}$, depending only on the size of Φ, such that, if $\Phi, \Phi^2, \ldots, \Phi^k$ do not show a modular coincidence, no Φ^ℓ $(\ell > k)$ does? If so, and if this k can be estimated (or determined) for any given Φ, Theorem 4.1 can be used to show that a given $\mathbf{V}$ does not consist of model sets. Therefore Theorems 4.2 and 4.4 would be obsolete in principle. On the other hand, if k is large, one would first try to use Theorems 4.2 and 4.4.

In general, the goal is to derive from the necessary and sufficient conditions of Theorem 4.1 resp. [LMS2] algorithms, which decide in finite time, whether any given lattice substitution system consist of model sets or not.

References

[AGS] R. Ammann, B. Grünbaum, G.C. Shephard: Aperiodic tiles. *Discr. & Comput. Geom.* **8** (1992) 1–25

[Baa] M. Baake: A guide to mathematical quasicrystals, in: *Quasicrystals*, eds J.–B. Suck, M. Schreiber & P. Häußler, Springer, Berlin (2002), pp 17–48

[BG] M. Baake, U. Grimm: A guide to quasicrystal literature, *Directions in Mathematical Quasicrystals*, eds. M. Baake & R.V. Moody, CRM Monograph Series Vol.13, AMS, Providence, RI (2000) pp 371–373

[BMP] M. Baake, R.V. Moody, P. Pleasants: Diffraction from visible lattice points and kth power free integers, *Discr. Math.* **221** (2000) 3–42, math.MG/9906132

[BSJ] M. Baake, M. Schlottmann, P.D. Jarvis: Quasiperiodic tilings with tenfold symmetry and equivalence with respect to local derivability, *J.Phys. A: Math. Gen.* **24** (1991) 4637–4654

[BS] M. Baake, B. Sing: Kolakoski-(3,1) is a (deformed) model set, Preprint, math.MG/0206098

[Ber] R. Berger: The undecidability of the domino problem, *Memoirs Amer. Math. Soc.* **66** (1966) 72–73

[BD] G. Bernuau, M. Duneau: Fourier analysis of deformed model sets, *Directions in Mathematical Quasicrystals*, eds. M. Baake & R.V. Moody, CRM Monograph Series Vol.13, AMS, Providence, RI (2000) pp 43–60

[Dan1] L. Danzer: 3-dimensional analogs of the planar Penrose tilings and quasicrystals, *Discr. Math.* **76** (1989) 1–7

[Dan2] L. Danzer: Inflation-species of planar tilings which are not of locally finite complexity, *Proc. of the Steklov Inst. of Math.* **239** (2002) 1–9

[Fre] D. Frettlöh: Nichtperiodische Inflationspflasterungen mit ganzzahligem Faktor, Ph D thesis, Dortmund (2002); http://eldorado.uni-dortmund.de:8080/FB1/ls2/forschung/2002/Frettloeh

[GS] B. Grünbaum, G.C. Shephard: *Tilings and Patterns*, Freeman, New York (1987)

[Kra] P. Kramer: Non-periodic central space filling with icosahedral symmetry using copies of seven elementary cells, *Acta Cryst.* **A 38** (1982) 257–264

[KN] P. Kramer, R. Neri: On periodic and non–periodic space fillings of $\mathbb{E}^m$ obtained by projection, *Acta Cryst.* **A 40** (1984) 580–587, and *Acta Cryst.* **A 41** (1985) 619 (Erratum)

[LM] Jeong-Yup Lee, R.V. Moody: Lattice substitution systems and model sets, *Discr. & Comput. Geom.* **25** (2001) 173–201, math.MG/0002019

[LMS1] Jeong-Yup Lee, R.V. Moody, B. Solomyak: Pure Point Dynamical and Diffraction Spectra, *Annales Henri Poincare* **3** (2002) 1003–1018

[LMS2] Jeong-Yup Lee, R.V. Moody, B. Solomyak: Consequences of Pure Point Diffraction Spectra for Multiset Substitution Systems, *Discr. & Comp. Geom.* **29** (2003) 525–560

[M] R.V. Moody: Meyer sets and their duals, *The Mathematics of Long-range Aperiodic Order*, ed R.V.Moody, NATO ASI Series C 489, Kluwer, Dordrecht (1996) pp 403–441

[M2] R.V. Moody: Model sets: a survey, *From Quasicrystals to More Complex Systems*, ed F. Axel & J.–P. Gazeau, Les Editions de Physique, Springer, Berlin (2000) pp 145–166, math.MG/0002020

[Pen] R. Penrose: The role of aesthetics in pure and applied mathematical research, *Bull. Inst. Math. & Appl.*, **10** (1974) 266–271

[Rad] C. Radin: The pinwheel tilings of the plane, *Annals of Math.* **139** (1997) pp 661–702

[Sch] M. Schlottmann: Generalized model sets and dynamical systems, *Directions in Mathematical Quasicrystals*, eds. M. Baake & R.V. Moody, CRM Monograph Series Vol.13, AMS, Providence, RI (2000) pp 143–159

[Sing] B. Sing: Kolakoski-(2m,2n) are limit-periodic model sets, *J. Math. Phys. 44* **2** (2003) 899-912 math-ph/0207037

[Sol1] B. Solomyak: Dynamics of self–similar tilings, *Ergod. Th. & Dyn. Syst.* **17** (1997) 695–738; Erratum, *Ergod. Th. & Dyn. Syst.* **19** (1999) 1685

[Sol2] B. Solomyak: Non–periodicity implies unique composition property for self–similar translationally finite tilings, *Discr. & Comput. Geom.* **20** (1998) 265–279

Institut für Mathematik und Informatik, Universität Greifswald, Jahnstr. 15a, 17487 Greifswald Germany

E-mail address: dirk.frettloeh@uni-greifswald.de

URL: http://paul.math-inf.uni-greifswald.de/~frettloe